

# *Upper bound for the span of $(s, 1)$ -total labelling of graphs*

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**N° 4816**

Mai 2003

THÈME 1



*rapport  
de recherche*



## Upper bound for the span of $(s, 1)$ -total labelling of graphs

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Thème 1 — Réseaux et systèmes  
Projet Mascotte

Rapport de recherche n° 4816 — Mai 2003 — 12 pages

**Abstract:** An  $(s, 1)$ -total labelling of a graph  $G$  is an assignment of integers to  $V(G) \cup E(G)$  such that: (i) any two adjacent vertices of  $G$  receive distinct integers, (ii) any two adjacent edges of  $G$  receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least  $s$  in absolute value. The *span* of a  $(s, 1)$ -total labelling is the maximum difference between two labels. The minimum span of a  $(s, 1)$ -total labelling of  $G$  is denoted by  $\lambda_s^T(G)$ .

In [4], it is conjectured that  $\lambda_s^T \leq \Delta + 2s - 1$ , where  $\Delta$  is the maximum degree of a vertex in a graph. This is an extension of the Total Colouring Conjecture. It is also shown that  $\lambda_s^T \leq 2\Delta + s - 1$  and  $\lambda_2^T(G) \leq 6$  if  $\Delta(G) \leq 3$  and  $\lambda_2^T(G) \leq 8$  if  $\Delta(G) \leq 4$ .

In this paper, we prove that  $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2\log(16s - 10)$ . The proof is an induction based on the maximal cut of a graph. We use the same technique to improve a little bit this result in the case of  $(2, 1)$ -total labelling. We prove that if  $\Delta(G) \geq 3$ , then  $\lambda_2^T(G) \leq 2\Delta(G)$  and that if  $\Delta(G) \geq 5$  is odd then  $\lambda_2^T(G) \leq 2\Delta(G) - 1$ .

**Key-words:**  $(s, 1)$ -total labelling, maximal cut

## Borne supérieure pour l'écart d'une coloration ( $s, 1$ )-totale

**Résumé :** Une *coloration* ( $s, 1$ )-totale d'un graphe  $G$  est une affectation d'entiers aux sommets et arêtes de  $G$  telle que: (i) deux sommets adjacents reçoivent des entiers distincts, (ii) deux arêtes adjacentes reçoivent des entiers distincts, et (iii) un sommet et une arête incidents reçoivent des entiers qui diffèrent d'au moins  $s$  en valeur absolue. L'*écart* d'une coloration ( $s, 1$ )-totale est la différence maximum entre deux entiers affectés. L'écart minimum d'une coloration ( $s, 1$ )-totale de  $G$  est noté  $\lambda_s^T(G)$ .

Dans [4], il est conjecturé que  $\lambda_s^T \leq \Delta + 2s - 1$ , avec  $\Delta$  le degré maximum d'un sommet du graphe. C'est une généralisation de la Conjecture de la Coloration Totale. Il est également montré que  $\lambda_s^T \leq 2\Delta + s - 1$  et  $\lambda_2^T(G) \leq 6$  si  $\Delta(G) \leq 3$  et  $\lambda_2^T(G) \leq 8$  si  $\Delta(G) \leq 4$ .

Nous montrons ici que  $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2\log(16s - 10)$ . La preuve est une récurrence basée sur la coupe maximale d'un graphe. Nous utilisons la même technique pour améliorer un peu cette borne dans le cas des colorations ( $2, 1$ )-totales. Nous prouvons que si  $\Delta(G) \geq 3$ , alors  $\lambda_2^T(G) \leq 2\Delta(G)$  et si  $\Delta(G) \geq 5$  est impair alors  $\lambda_2^T(G) \leq 2\Delta(G) - 1$ .

**Mots-clés :** coloration ( $s, 1$ )-totale, coupe maximale

## 1 Introduction

Let  $G$  be a graph. The degree of a vertex  $v$  is denoted by  $d_G(v)$  or  $d(v)$  if  $G$  is clearly understood. The maximum degree of  $G$  is denoted by  $\Delta(G)$ . An  $(s, 1)$ -total labelling of a graph  $G$  is an assignment of integers to  $V(G) \cup E(G)$  such that: (i) any two adjacent vertices of  $G$  receive distinct integers, (ii) any two adjacent edges of  $G$  receive distinct integers, and (iii) incident vertex and edge receive integers that differ by at least  $s$  in absolute value. The *span* of an  $(s, 1)$ -total labelling is the maximum difference between two labels. The minimum span of a  $(s, 1)$ -total labelling of  $G$  is denoted by  $\lambda_s^T(G)$ . Note that a  $(1, 1)$ -total labelling is a total colouring and that  $\lambda_1^T(G) = \chi^T - 1$  where  $\chi^T$  is the total colouring number.

An  $(s, 1)$ -total labelling of a graph  $G$  corresponds to an  $L(s, 1)$ -labelling of its incidence graph  $I(G)$  which is the bipartite graph defined as follows :  $V(I(G)) = V(G) \cup E(G)$  and  $ve \in E(I(G))$  if and only if  $v \in V(G)$ ,  $e \in E(G)$  and  $v$  and  $e$  are incident.  $L(2, 1)$ -labellings were first introduced in Griggs and Yeh [3] and  $L(s, 1)$ -labelling have been studied for several class of graphs, for example chordal graphs [1] or planar graphs [5]. The  $(2, 1)$ -total labellings of graphs were first studied by Whittlesey, Georges and Mauro [7] as  $L(2, 1)$ -labellings of incidence graphs. In [4], Havet and Yu investigate  $(s, 1)$ -total labelling for any  $s$ . They derive from Brooks and Vizing's Theorems that  $\lambda_s^T \leq 2\Delta + s - 1$ . Generalizing the Total Colouring Conjecture, they conjecture the following :

**Conjecture 1 (Havet and Yu [4])**

$$\lambda_s^T \leq \Delta + 2s - 1$$

By the previous result, it suffices to prove the conjecture for  $s < \Delta$ . Rosenfeld [6] established that  $\lambda_1^T(G) \leq 4$  if  $\Delta(G) \leq 3$ . Havet and Yu completed the proof of the conjecture for  $\Delta \leq 3$  by proving that  $\lambda_2^T(G) \leq 6$  if  $\Delta(G) \leq 3$ .

In this paper, we improve Havet and Yu's upper bound by showing  $\lambda_s^T \leq 2\Delta - \log(\Delta + 2) + s - 1 + 2\log(16s - 8)$ . The proof is an induction based on the maximal cut of a graph. The idea and tools are presented Section 2 and the proof is given Section 3. Finally using the same technique to improve a little bit the result in the case of  $(2, 1)$ -total labelling. We prove that if  $\Delta(G) \geq 3$ , then  $\lambda_2^T(G) \leq 2\Delta(G)$  this generalizes results of Havet and Yu [4] who proved it for  $\Delta \in \{3, 4\}$ . Furthermore, we show that if  $\Delta(G) \geq 5$  is odd then  $\lambda_2^T(G) \leq 2\Delta(G) - 1$ .

## 2 The tools and the idea

**Definition 1** A *cut*  $[A, B]$  of a graph  $G$  is a set of two induced subgraphs  $A$  and  $B$  of  $G$  such that  $(V(A), V(B))$  is a partition of  $V(G)$ . The bipartite graph  $(A, B)$  is the graph with vertex set  $V(G)$  and edge set  $E(G) \setminus (E(A) \cup E(B))$ . The edges of  $(A, B)$  are called the *cut edges*. A *maximum cut*  $[A, B]$  is a cut with maximum number of cut edges.

**Lemma 1** Let  $G$  be a graph with maximum degree  $2k + 1$ . Then a maximum cut  $[A, B]$  satisfies  $\Delta(A) \leq k$  and  $\Delta(B) \leq k$ .

**Proof.** Consider a maximum cut  $[A, B]$ .  $B$  contains no vertex  $b$  of degree greater than  $k$  otherwise  $[A + b, B - b]$  is a cut with strictly more cut edges. Analogously  $A$  has no vertex of degree greater than  $k$ .  $\square$

**Lemma 2** *Let  $G$  be a graph with maximum degree  $2k$ . Then  $G$  has a cut  $[A, B]$  such that  $\Delta(A) \leq k$  and  $\Delta(B) \leq k$ .*

**Proof.** Consider a maximum cut  $[A, B]$  which minimizes the number of vertices with degree  $k$  in  $A$ . As in the proof of Lemma 1,  $A$  and  $B$  contain no vertex of degree greater than  $k$ . Moreover  $A$  has no vertex  $a$  of degree  $k$  otherwise  $[A - a, B + a]$  is a cut with the same number of cut edges as  $[A, B]$  and one vertex less of degree  $k$  in the first subgraph.  $\square$

**Lemma 3** *Let  $G$  be a bipartite graph with maximum degree  $\Delta$ . There is an edge colouring  $c$  of  $G$  in  $[1, \Delta]$  such that  $c(e) \geq i$  only if it is incident to a vertex of degree at least  $i$ .*

**Proof.** Let us prove it by induction on  $\Delta$ , the result holding trivially when  $\Delta = 0$ . Consider now a graph with maximum degree  $\Delta \geq 1$ . By König's theorem, it admits an edge colouring  $c_1$  in  $[1, \Delta]$ . Let  $M$  be the set of edges coloured  $\Delta$  incident to a vertex of degree  $\Delta$ . Consider  $G'$  the graph obtained from  $G$  by removing  $M$ . Since every vertex of degree  $\Delta$  is adjacent to an edge of  $M$ ,  $\Delta(G') = \Delta - 1$ . Then by induction  $G'$  has an edge colouring  $c$  of  $G$  in  $[1, \Delta - 1]$  such that  $c(e) \geq i$  only if it is incident to a vertex of degree at least  $i$ . Extending  $c$  into an edge colouring of  $G$  in  $[1, \Delta]$  by colouring the edges of  $M$  with  $\Delta$ , we obtain the result.  $\square$

**Definition 2** Let  $G$  be a graph. A *list assignment*  $L$  is an assignment of a set  $L(v)$  of integers to every vertex  $v$  of  $G$ . The graph  $G$  is  *$L$ -colourable* if it admits an application  $c$  called  *$L$ -colouring* from its vertex set into the set of integers such that for any vertex  $v$ ,  $c(v) \in L(v)$  and for any edge  $(u, v)$ ,  $c(u) \neq c(v)$ .

Let  $k$  be a non-negative integer. A  *$k$ -list assignment* is an assignment  $L$  such that  $|L(v)| = k$  for every vertex  $v$ . A graph is  *$k$ -choosable* if it is  $L$ -colourable for any  $k$ -list assignment  $L$ .

Let  $v$  be a vertex of  $G$ . A  *$(d, v)$ -list assignment* of  $G$  is a list assignment  $L$  such that  $|L(u)| = d(u)$  if  $u \neq v$  and  $|L(v)| = d(v) + 1$ . We say that  $G$  is  *$(d, v)$ -choosable* if it is  $L$ -colourable for any  $(d, v)$ -list assignment  $L$ .

**Proposition 1** *Let  $G$  be a connected graph and  $v \in V(G)$ . Then  $G$  is  $(d, v)$ -choosable.*

**Proof.** There is an ordering  $v_1, v_2, \dots, v_n$  of the vertices of the graph such for  $i < n$  the vertex  $v_i$  has a neighbour in  $\{v_j, i < j \leq n\}$ . Hence by a greedy algorithm, one can find an  $L$ -colouring of  $G$  for any  $(d, v)$ -list assignment  $L$ .  $\square$

Using this proposition, we strengthen Havet and Yu result [4] stating that  $\lambda_s^T(G) \leq 2\Delta + s - 1$ .

**Lemma 4** *Let  $G$  be a graph with maximum degree  $\Delta \leq k$ .  $G$  admits a  $(s, 1)$ -total labelling in  $[0, 2k + s - 1]$  such that a vertex  $v$  is assigned a label in  $[0, d(v)]$  and an edge a label in  $[k + s - 1, 2k + s - 1]$ .*

**Proof.** Obviously it suffices to prove it when  $G$  is connected.

By Vizing's Theorem, there is an edge colouring  $c'$  of  $G$  with colours in  $[k + s - 1, 2k + s - 1]$ . Let  $v$  be a vertex of  $G$ . Free to permute the colours of  $c'$ , we may assume that for every edge incident to  $v$ ,  $c'(v) \geq k + s$ . Let  $L$  be the  $(d, v)$ -list assignment with  $L(u) = [0, d(u) - 1]$  if  $u \neq v$  and  $L(v) = [0, d(v)]$ . By Proposition 1,  $G$  has an  $L$ -colouring. The union of  $c$  and  $c'$  is an  $(s, 1)$  total labelling of  $G$ .

Indeed for every edge  $e = xy$ , if  $x \neq v$  then  $c(x) \leq k - 1 \leq c'(e) - s$  and if  $x = v$  then  $c(v) \leq k \leq c'(e) - s$ .  $\square$

Analogously, we have the following lemma:

**Lemma 5** *Let  $G$  be a graph with maximum degree  $\Delta \leq k$ .  $G$  admits a  $(s, 1)$ -total labelling in  $[0, 2k + s - 1]$  such that an edge is assigned a label in  $[0, k]$  and a vertex  $v$  a label in  $[k + s - 1, k + s - 1 + d(v)]$ .*

**Theorem 1 (Galvin [2])** *Every bipartite graph  $G$  is  $\Delta(G)$ -edge choosable.*

The idea of the results is to consider a suitable maximum cut of  $G$  given by Lemma 1 or 2 and to label edge and vertices of  $A$  and  $B$  with Lemma 4 or induction hypothesis and Lemma 5 respectively and then to label the bipartite graph  $(A, B)$  using Lemma 3. Some few relabellings are then necessary to obtain the desired  $(s, 1)$ -total labelling. Theorem 1 is used for some of the relabellings.

### 3 Main result

The aim of this section is to prove the following theorem :

**Theorem 2** *For any  $s \geq 1$ ,*

$$\lambda_s^T \leq 2\Delta - 2\log(\Delta + 2) + 2\log(16s - 8) + s - 1$$

In order to prove this theorem, we prove by induction a stronger result.

Let  $G$  be a graph with maximal degree  $\Delta$ . An  $(s, 1)$ -total labelling in  $[0, p]$  is a  *$s$ -good labelling* if each vertex is assigned a label in  $[0, \Delta + s - 1]$ .

**Theorem 3** *Let  $G$  be a graph with maximal degree  $\Delta$ . Then  $G$  has a  $s$ -good labelling in  $[0, 2\Delta - 2\log(\Delta + 2) + 2\log(16s - 8) + s - 1]$ .*

The idea is to prove this result by induction. Note that Lemma 5 give the result for small value of  $\Delta$ . We will now give two Lemmas allowing us to do an induction step.

**Lemma 6** *Let  $k \geq \max(i + 2s - 1, 2i + 6s - 5)$ . If every graph of maximal degree  $k$  admits a  $s$ -good labelling in  $[0, 2k - i]$  then every graph  $G$  of maximal degree  $\Delta = 2k + 2$  admits a  $s$ -good labelling in  $[0, 2\Delta - i - 2]$ .*

**Proof.** According to Lemma 2 there is a cut  $[A, B]$  of  $G$  such that  $\Delta(A) \leq k$  and  $\Delta(B) \leq k + 1$ . Thus by hypothesis, there is a  $s$ -good labelling of  $A$  in  $[0, 2k - i]$ . And by Lemma 5, there is an  $(s, 1)$ -total labelling of  $B$  such that vertices are labelled in  $[k + s, k + s + d_B(v)]$  and edges in  $[0, k + 1]$ .

By Lemma 3, label the edges of  $(A, B)$  with  $[2k - i + 1, 4k - i + 2]$  so that an edge is labelled  $4k - i + 3 - l$  only if it is incident to a vertex of degree at least  $l$  in  $(A, B)$ .

The obtained labelling is not yet an  $(s, 1)$ -total labelling. Indeed for  $j \in [0, i + 2s - 1]$ , edges  $(a, b)$  labelled  $2k - i + 1 + j$  when  $b$  is labelled in  $[2k - i + j - s + 2, 2k - i + j + s]$  violate the constraints. Hence they must be relabelled.

Let us consider the bipartite graph induced by such edges. It has degree at most  $i + 2s$ . We want to relabel the edges with labels in  $[k + 2s - 2, 2k - i]$ . According to Theorem 1, we need to find a list of  $i + 2s$  available labels for each edge. Let  $(a, b)$  be an edge labelled  $2k - i + 1 + j$  with  $b$  labelled in  $[2k - i + j - s + 2, 2k - i + j + s]$ . Then  $d_B(b) \geq k - i + j - 2s + 2$ . So  $b$  has degree at most  $k + i - j + 2s$  in  $(A, B)$ . But by construction  $(a, b)$  is incident to a vertex of degree at least  $2k + 2 - j$  in  $(A, B)$ . Since  $k \geq i + 2s - 1$  then this vertex is  $a$  and  $d_A(a) \leq j$ . So at most  $j$  labels of  $[k + 2s - 2, 2k - i]$  are forbidden because of the edges of  $A$  incident to  $a$ . Moreover at most  $2s - j - 2$  labels of  $[k + 2s - 2, 2k - i]$  are forbidden because of  $b$  (those of  $[2k - i + j - 2s + 3, 2k - i]$ ). Hence at most  $2s - 2$  labels of  $[k + 2s - 2, 2k - i]$  are forbidden. So because  $k \geq 2i + 6s - 5$ , at least  $k - i - 2s + 3 - (2s - 2) \geq i + 2s$  labels available on  $(a, b)$ .

Since the labels of the vertices are in  $[0, 2k + 1 + s]$ , we have a  $s$ -good labelling of  $G$  in  $[0, 4k - i + 2]$ .  $\square$

**Lemma 7** *Let  $k \geq \max(i + 4s - 1, 2i + 6s - 3)$ . If every graph of maximal degree  $k$  admits a  $s$ -good labelling in  $[0, 2k - i]$  then every graph  $G$  of maximal degree  $\Delta = 2k + 1$  admits a  $s$ -good labelling in  $[0, 2\Delta - i - 2]$ .*

**Proof.** Let  $[A, B]$  be a maximum cut of  $G$ . Then  $\Delta(A) \leq k$  and  $\Delta(B) \leq k$ . Thus by hypothesis, there is a  $s$ -good labelling of  $A$  in  $[0, 2k - i]$ . And by Lemma 5, there is an  $(s, 1)$ -total labelling of  $B$  such that vertices are labelled in  $[k + s, k + s + d_B(v)]$  and edges in  $[1, k]$ .

By Lemma 3, label the edges of  $(A, B)$  with  $[2k - i, 4k - i]$  so that an edge is labelled  $4k - i + 1 - l$  only if it is incident to a vertex of degree at least  $l$  in  $(A, B)$ .

There are two types of edges of  $(A, B)$  violating a constraint of an  $(s, 1)$ -total labelling :

- (1) edges  $(a, b)$  labelled  $2k - i + j$  while  $b$  is labelled in  $[2k - i + j - s + 1, 2k - i + j + s - 1]$  for some  $j \in [0, i + 2s - 1]$ ;
- (2) edges  $(a, b)$  labelled  $2k - i$  with  $a$  incident to an edge (of  $A$ ) labelled  $2k - i$ .



Let us first relabel the edges of type (1) with labels in  $[k + 2s - 1, 2k - i - 1]$ . Let us consider the bipartite graph induced by them. It has degree at most  $i + 2s$ . According to Theorem 1, we need to find a list of  $i + 2s$  available labels for each edge. Let  $(a, b)$  be an edge labelled  $2k - i + j$  with  $b$  labelled in  $[2k - i + j - s + 1, 2k - i + j + s - 1]$ . Then  $d_B(b) \geq k - i + j - 2s + 1$ . So  $b$  has degree at most  $k + i - j + 2s$  in  $(A, B)$ . But by construction  $(a, b)$  is incident to a vertex of degree at least  $2k + 1 - j$  in  $(A, B)$ . Since  $k \geq i + 2s$ , this vertex is  $a$  and  $d_A(a) \leq j$ . So at most  $j$  labels are forbidden because of the edges of  $A$  incident to  $a$  and at most  $2s - j - 2$  are forbidden because of  $b$  (those of  $[2k - i + j - 2s + 2, 2k - i - 1]$ ). Hence at most  $2s - 2$  labels of  $[k + 2s - 1, 2k - i - 1]$  are forbidden. So since  $k \geq 2i + 6s - 3$ , there are at least  $k - i - 2s + 1 - (2s - 2) \geq i + 2s$  labels available on  $(a, b)$ .

Let us now relabel the edges of type (2). Since  $a$  is incident to an edge of  $A$ , it has degree less than  $2k + 1$  in  $(A, B)$ . Hence  $b$  has degree  $2k + 1$  in  $(A, B)$  and thus is isolated in  $B$ . In particular  $b$  was not incident to an edge of type (1). Let  $l(a)$  be the label of  $a$ . There is a label in  $[0, k + 2s - 1] \setminus [l(a) - s + 1, l(a) + s - 1]$  that is not assigned to any edge of  $A$  incident to  $a$ . Relabel  $(a, b)$  with  $l$ . Since  $l + s \leq k + 3s - 1 \leq 2k - i - s$ , we can relabel  $b$  with  $k + 3s - 1$ .

Since the labels of the vertices are in  $[0, 2k + s]$  we have a good labelling of  $G$  in  $[0, 4k - i]$ .

□

**Proof of Theorem 3.** Set  $c_s = 2\log(16s - 8) + s - 1$ . If  $\Delta \leq 16s - 10$ , then we have the result by Lemma 4. Suppose now that  $G$  is a graph with maximal degree  $\Delta \geq 16s - 9$ .

If  $\Delta$  is even, set  $\Delta = 2k + 2$ . By induction hypothesis  $\lambda_s^T(H) \leq 2k - 2\log(k + 2) + c_s$ . And setting  $i = 2\log(k + 2) - c_s$ , we have  $k \geq \max(i + 2s - 2, 2i + 6s - 5)$ . Hence by Lemma 6,  $\lambda_s^T(G) \leq 2\Delta - 2\log(k + 2) + c_s - 2$ . Since  $\log(k + 2) + 1 = \log(2k + 4) = \log(\Delta + 2)$ . We obtain  $\lambda_s^T(G) \leq 2\Delta - 2\log(\Delta + 2) + c_s$ .

In the same way, we have the result if  $\Delta$  is odd.

□

## 4 Better bounds when $s = 2$

### 4.1 Upper bound $2\Delta$

**Theorem 4** *If  $\Delta(G)$  is odd and at least 5 then  $G$  has a 2-good labelling in  $[0, 2\Delta(G)]$ .*

**Proof.** Set  $\Delta(G) = 2k + 1$ . Consider a maximum cut  $[A, B]$  of  $G$ . Then  $\Delta(A) \leq k$  and  $\Delta(B) \leq k$ .

Thus by Lemmas 4 and 5, one may label  $A$  and  $B$  in  $[0, 2k + 1]$  such that a vertex  $v$  in  $A$  receives a label in  $[0, d_A(v)]$  (resp.  $[k + 1, k + 1 + d_B(v)]$ ) and edges labels in  $[k + 1, 2k + 1]$  (resp.  $[0, k]$ ).

Now by Lemma 3, label the edges of  $(A, B)$  in  $[2k + 2, 4k + 2]$  such that an edge is assigned  $2k + 2$  only if it is adjacent to a vertex with degree  $2k + 1$  in  $(A, B)$  and so an isolated vertex in  $A$  or  $B$ .

The label of an edge  $(a, b)$  of  $(A, B)$  fullfill the constraints of a  $(2, 1)$ -total labelling unless it is labelled  $2k + 2$  and  $b$  is labelled  $2k + 1$ . But in this case,  $a$  is an isolated vertex of  $A$  and thus labelled 0. So we may relabel  $(a, b)$  with  $k + 1$ . This is possible since  $k \geq 2$  so  $(2k + 1) - (k + 1) \geq 2$ .

Since the vertices are labelled in  $[0, 2k + 1]$  we have a 2-good labelling.  $\square$

The proof of Theorem 4 does not work when  $\Delta = 3$ . However, we give an alternative proof of a result of Havet and Yu [4] asserting that a graph with maximum degree 3 has a  $(2, 1)$ -total labelling in  $[0, 6]$ .

**Theorem 5** *If  $\Delta(G) \leq 3$  then  $\lambda_2^T(G) \leq 6$ .*

**Proof.** Let  $[V_1, V_2]$  a maximal cut of  $G$ . Easily  $\Delta(V_i) \leq 1$ .

For  $i = 1, 2$ , let  $S_i$  (resp.  $T_i$ ) be the set of isolated vertices (resp. vertices with degree 1) in  $G_i$ .

Label the edges of  $V_1$  (resp.  $V_2$ ) with 3 (resp. 0) and their endvertices with 0 and 1 (resp. 2 and 3). Label the vertices of  $S_2$  with 2.

By König's Theorem, there is a 3-edge colouring of  $(V_1, V_2)$  with colours  $a$ ,  $b$  and  $c$ . For each  $a$ -coloured edge  $(u, v)$  with  $u \in G_1$  do the following :

- If  $u \in S_1$  and  $v \in S_2$ , assign 4 to  $(u, v)$  and 0 to  $u$ .
- If  $u \in T_1$  and  $v \in S_2$ , assign 4 to  $(u, v)$ .
- If  $u \in S_1$ ,  $v \in T_2$  and  $v$  is labelled 2 then assign 4 to  $(u, v)$  and 0 to  $u$ .

At this stage the vertices of  $S_1$  whose incident  $a$ -coloured edge has an end in  $T_2$  labelled 3 are not yet coloured. We will label them one after another doing the following algorithm :

- (1) If there is a vertex  $y \in T_2$  that is adjacent to two non labelled vertices  $x$  and  $z$  (of  $S_1$ ), assign 0 to  $x$  and  $z$ , 3 to  $(x, y)$ , 4 to  $(y, z)$  and relabel  $y$  with 6. Go to (1).
- (2) If there is a vertex  $y \in T_2$  that is adjacent to a non-labelled vertex  $x$  and a labelled vertex  $z \in S_1$ , then  $z$  is labelled 0 and there is an integer  $l$  in  $\{2, 3, 4\}$  that label no edge incident to  $z$ . Then assign 0 to  $x$ ,  $l$  to  $(y, z)$ , an integer of  $\{2, 3, 4\} \setminus \{l\}$  to  $(x, y)$  and relabel  $y$  with 6. Go to (2).
- (3) If there is a vertex  $y \in T_2$  that is adjacent to a non-labelled vertex  $x$  and a vertex  $z \in T_1$ . Let  $e$  be the edge of  $B$  incident to  $z$  and distinct from  $(y, z)$ .

If  $e$  is not labelled yet then assign 4 to  $(y, z)$ , 3 to  $(x, y)$  and 0 to  $x$ . Relabel  $y$  with 6. Go to (3).

Otherwise  $e$  is already labelled with 4. Let  $a$  be the label of  $z$ . Assign 6 to  $(y, z)$ , 4 to  $(x, y)$  and  $a$  to  $x$ . Relabel  $y$  with the integer of  $\{0, 1\} \setminus \{a\}$ . Go to (3).

Let  $E'$  be the set of non labelled edges. It induces a bipartite graph with maximum degree 2. And the vertices incident to edges of  $E'$  are labelled in  $[0, 3]$ .

By König's theorem,  $E'$  can be two coloured with label 5 and 6. It is easy to see that we have a  $(2, 1)$ -total labelling of  $G$ .  $\square$

**Remark 1** The  $(2, 1)$ -total labelling obtained by such a proof is really different from the one obtained by the proof of Havet and Yu [4].

**Theorem 6** *If  $\Delta(G)$  is even and at least 6 then  $G$  has a 2-good labelling in  $[0, 2\Delta(G)]$ .*

**Proof.** Set  $\Delta = 2k$ . Consider a cut  $[A, B]$  as in Lemma 2. Following Lemma 5, label  $A$  such that a vertex  $v$  receives a label in  $[k + 1, k + 1 + d_A(v)]$  and an edge a label in  $[1, k]$ . Following Lemma 4, label  $B$  such that a vertex  $v$  receives a label in  $[0, d_B(v)]$  and an edge a label in  $[k + 1, 2k + 1]$ .

Now by Lemma 3, label the edges of  $(A, B)$  in  $[2k + 1, 4k]$  such that an edge is assigned  $2k + 1$  only if it is adjacent to a vertex with degree  $2k$  in  $(A, B)$  and so an isolated vertex in  $A$  or  $B$ .

The label of an edge  $(a, b)$  of  $(A, B)$  fullfill the constraints of a  $(2, 1)$ -total labelling unless  $(a, b)$  is labelled  $2k + 1$  and 1)  $a$  is labelled  $2k$  or 2)  $b$  is incident to an edge of  $B$  labelled  $2k + 1$ . Thus we need some relabelling.

1) If  $a$  is labelled  $2k$ , then  $a$  is not isolated in  $A$ . Thus  $b$  is isolated in  $B$ . Then relabel  $(a, b)$  with 0 and  $b$  with 2.

2) If  $b$  is incident to an edge  $(b, b')$  of  $B$  which is labelled  $2k + 1$ , then  $b$  is not isolated in  $B$ . Thus  $a$  is isolated in  $A$ . In particular such an edge is disjoint from any edge of type 1). Let  $l(b)$  be the label assigned to  $b$ . If  $l(b) \geq 2$  then relabel  $(a, b)$  with 0. If  $l(b) \leq 1$  then relabel  $(a, b)$  with 3 and  $a$  with 5 if  $k = 3$ . This is valid since  $k \geq 3$ .

In such a  $(2, 1)$ -total labelling a vertex is assigned an integer in  $[0, 2k]$ , so we have a 2-good labelling.  $\square$

One can extend Theorem 6 for  $\Delta = 4$ . This strengthen a result of Havet and Yu [4] stating that  $\lambda_2(G) \leq 8$  if  $\Delta \leq 4$ .

**Theorem 7** *If  $\Delta(G) = 4$  then  $G$  has a 2-good labelling in  $[0, 8]$ .*

**Proof.** By Lemma 2,  $G$  has a cut  $[A, B]$  such that  $\Delta(A) \leq 1$  and  $\Delta(B) \leq 2$ .

Label the vertices of  $A$  with  $\{0, 1\}$  and its edges with  $\{3\}$  such that the isolated vertices of  $A$  receive 0.

Label the vertices and edges of  $B$  which do not lay on odd cycle of  $B$  as follows :

- (i) the isolated vertices of  $B$  are labelled 3;
- (ii) The vertices and edges of an even cycle or a path are labelled alternatively 3 and 4 and 0 and 1 respectively.

According to Lemma 3, label the edges of  $(A, B)$  with  $[5, 8]$  so that an edge assigned 5 is incident to a vertex of degree 4 in  $(A, B)$  which are isolated vertices in  $A$  or  $B$ .

Some constraints are violated each time an edge  $(a, b)$  of  $(A, B)$  is labelled 5 and  $a$  is labelled 4. But in that case,  $a$  is not isolated in  $A$  thus  $b$  is isolated in  $B$  and so is labelled 0. Then relabel  $(a, b)$  with 2.

At this stage, it remains to assign labels to vertices and edges of odd cycles of  $G$ .

Let  $C = (b_0, b_1, \dots, b_{2p}, b_0)$  be an odd cycle of  $B$ . Then two consecutives vertices, say  $b_0$  and  $b_1$  are either both incident to an edge labelled 5 or both non incident to an edge labelled 5. Then for  $1 \leq i \leq p$ , label  $b_{2i-1}$  with 3,  $b_{2i}$  with 4,  $(b_{2i-1}, b_{2i})$  with 1 and  $(b_{2i}, b_{2i+1})$  with 0. And label  $b_0$  with 2.

If  $b_0$  and  $b_1$  are non incident to an edge labelled 5 then label  $(b_0, b_1)$  with 5. Otherwise there is a label  $l \in [6, 8]$  such that both  $b_0$  and  $b_1$  are incident to no edge labelled  $l$ . Label  $(b_0, b_1)$  with  $l$ .

Since the vertices are labelled in  $[0, 4]$ , we have a 2-good labelling of  $G$  in  $[0, 8]$ .  $\square$

**Corollary 1**  $\lambda_2^T(G) \leq 2\Delta - 2\log(\Delta + 2) + 8$

## 4.2 Upper bound $2\Delta - 1$ for odd $\Delta$

**Theorem 8** *If  $\Delta(G)$  is odd and at least 7 then  $G$  has a 2-good labelling in  $[0, 2\Delta(G) - 1]$ .*

**Proof.** Set  $\Delta(G) = 2k + 1$ . Consider a maximum cut  $[A, B]$  of  $G$ . Then  $\Delta(A) \leq k$  and  $\Delta(B) \leq k$ .

Following Lemma 4, label  $A$  such that each vertex  $v$  of  $A$  is assigned a label in  $[0, d_A(v)]$  and each edge  $e$  a label in  $[k + 1, 2k + 1]$ .

Following Lemma 5, label  $B$  such that each vertex  $v$  of  $B$  is assigned a label in  $[k + 1, k + 1 + d_B(v)]$  and each edge  $e$  a label in  $[0, k]$ .

By Lemma 3, label the edges of  $(A, B)$  with  $[2k + 1, 4k + 1]$  so that an edge is labelled  $4k + 2 - i$  only if it is incident to a vertex of degree  $i$  in  $(A, B)$ .

This labelling may violate some constraints of a  $(2, 1)$ -total labelling in the following cases :

- (1) a vertex  $b \in B$  labelled  $2k$  or  $2k + 1$  is incident to an edge  $(a, b)$  of  $(A, B)$  labelled  $2k + 1$ ;
- (2) a vertex  $b \in B$  labelled  $2k + 1$  is incident to an edge  $(a, b)$  of  $(A, B)$  labelled  $2k + 2$ ;
- (3) a vertex  $a \in A$  is incident to two edges labelled  $2k + 1$  one  $(a, a')$  in  $A$  and one  $(a, b)$  in  $(A, B)$ ;

Therefore, we need to proceed to the following corresponding relabelling :

- (1) Since  $k \geq 2$ , then  $2k > k + 1$  so  $b$  is not isolated in  $B$ . Thus the vertex  $a$  is isolated in  $A$  and labelled 0. Then relabel  $(a, b)$  with  $k$ .

- (2) The vertex  $b$  is labelled  $2k + 1$  and so  $d_B(b) = k \geq 2$ . Hence  $b$  has degree less than  $2k$  in  $(A, B)$  and  $a$  has degree at least  $2k$  in  $(A, B)$ . So  $a$  has degree at most 1 in  $A$  and thus is labelled 0 or 1. One of the two integers  $k + 1$  and  $k + 2$  does not label the (possible) edge incident to  $a$  in  $A$ . Then relabel  $(a, b)$  with  $l$ . This is valid since  $k \geq 3$ .
- (3) Since  $a$  is not isolated in  $A$ , then  $b$  is isolated in  $B$  and thus labelled  $k + 1$ . If  $a$  is labelled 0 or 1 then relabel  $(a, b)$  with 3 and  $b$  with 5 if  $k = 3$ . Again this is valid since  $k \geq 3$ . If  $a$  is labelled in  $[2, k + 1]$  then relabel  $(a, b)$  with 0.

□

The last two relabelling of the previous proof are not valid if  $k = 2$ . Hence, to get the result when  $\Delta = 5$ , we need some extra arguments :

**Theorem 9** *If  $\Delta(G) = 5$  then  $\lambda_2^T(G) \leq 9$ .*

**Proof.**

Let  $[A, B]$  be a maximum cut of  $G$ . Then  $\Delta(A) \leq 2$  and  $\Delta(B) \leq 2$ .

We need a more careful labelling of  $A$  than in Theorem 8. Let  $C$  be a component of  $A$ . If  $C$  is not an odd cycle then, following Lemma 4, label  $C$  such that each vertex  $v$  is assigned a label in  $[0, d_A(v)]$  and each edge  $e$  a label in  $[3, 4]$ . If  $C$  is an odd cycle  $(a_1, a_2, \dots, a_{2p+1}, a_1)$  then for  $1 \leq i \leq p$ , label  $a_{2i-1}$  with 0,  $(a_{2i-1}, a_{2i})$  with 3,  $a_{2i-1}$  with 1, and  $(a_{2i}, a_{2i+1})$  with 4. Label  $a_{2p+1}$  with 2 and  $(a_{2p+1}, a_1)$  with 5. Note that in that case a vertex labelled 1 in  $A$  is not incident to an edge labelled 5.

Following Lemma 5, label  $B$  such that each vertex  $v$  of  $B$  is assigned a label in  $[3, 3 + d_B(v)]$  and each edge  $e$  a label in  $[0, 2]$  with  $B_e$ .

By Lemma 3, label the edges of  $(A, B)$  with  $[5, 9]$  so that an edge is labelled  $10 - i$  only if it is incident to a vertex of degree at least  $i$  in  $(A, B)$ .

This labelling may violate the constraints of a  $(2, 1)$ -total labelling in the same cases as in Theorem 8 :

- (1) a vertex  $b \in B$  labelled 4 or 5 is incident to an edge  $(a, b)$  of  $(A, B)$  labelled 5;
- (2) a vertex  $b \in B$  labelled 5 is incident to an edge  $(a, b)$  of  $(A, B)$  labelled 6;
- (3) a vertex  $a \in A$  is incident to two edges labelled 5 one  $(a, a')$  in  $A$  and one  $(a, b)$  in  $(A, B)$ ;

Therefore, we need to proceed to the following corresponding relabelling :

- (1) As in Theorem 8, relabel  $(a, b)$  with 2.
- (2) The vertex  $b$  is labelled 5 and so  $d_B(b) = 2$ . Hence  $b$  has degree less than 6 in  $(A, B)$  and  $a$  has degree at least 6 in  $(A, B)$ . So  $a$  has degree at most 1 in  $A$  and thus is labelled 0 or 1. Relabel  $(a, b)$  with 3. This may violate a constraint if the edge  $(a, a')$  in  $A$  incident to  $a$  is also labelled 3. If  $a'$  is incident to no edge labelled 4 then relabel  $(a, a')$  with 4. Otherwise  $d_{(A, B)}(a') \leq 2$ . Thus there is a label  $l \in [5, 7]$  that labels no edge incident to  $a$  or  $a'$  (since  $(a, b)$  is now labelled 3). Relabel  $(a, a')$  with  $l$ .

- (3) Since  $a$  is not isolated in  $A$ , then  $b$  is isolated in  $B$  and thus labelled 3. Moreover, the vertex  $a$  is labelled either 0 or 2 because no vertex of  $A$  labelled 1 is incident to an edge of  $A$  labelled 5. If  $a$  is labelled 0 or then relabel  $(a, b)$  with 2 and  $b$  with 4. If  $a$  is labelled 2 then relabel  $(a, b)$  with 0.

□

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INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)  
<http://www.inria.fr>  
ISSN 0249-6399